Journal of Pure and Applied Mathematics: Advances and Applications
Volume 10, Number 1, 2013, Pages 103-113

AN EXTENSION OF FUGLEDE-PUTNAM THEOREM FOR CLASS $A(t, t)$ OPERATORS

F. LOMBARKIA and A. BACHIR

Department of Mathematics
Faculty of Science
University of Batna
05000, Batna
Algeria
e-mail: lombarkiafarida@yahoo.fr

Department of Mathematics
Faculty of Science
King Khalid University
Abha, P. O. Box 9004
Saudi Arabia
e-mail: bachir_ahmed@hotmail.com

Abstract

The familiar Fuglede-Putnam theorem asserts that if A and B are normal operators and $AX = XB$ for some bounded operator X, then $A^*X = XB^*$. In this paper, the hypothesis on A and B are relaxed by using a Hilbert Schmidt operator X: if A is class $A(t, t)$ and B^* is invertible class $A(t, t)$ such that $AX = XB$ for some Hilbert Schmidt operator X, then $A^*X = XB^*$. As a consequence of this result, we obtain that the range of the generalized derivation induced by this class of operators is orthogonal to its kernel.

2010 Mathematics Subject Classification: 47B47, 47A30, 47B20.

Keywords and phrases: Hilbert Schmidt class, Fuglede-Putnam theorem, hyponormal operator, w-hyponormal operator, class $A(s, t)$ operators, orthogonality.

Received February 5, 2013; Revised March 12, 2013

1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space, and let $B(\mathcal{H})$, C_2 and C_1 denote the algebra of all bounded linear operators on $\mathcal{H}$, the Hilbert Schmidt class, and the trace class in $B(\mathcal{H})$, respectively. It is well known that $C_2(\mathcal{H})$ and $C_1(\mathcal{H})$ each form a two-sided $^*-$ideal in $B(\mathcal{H})$ and $C_2(\mathcal{H})$ is itself a Hilbert space with the inner product

$$\langle X, Y\rangle = \sum \langle Xe_i, Ye_i\rangle = tr(Y^*X) = tr(XY^*),$$

where $\{e_i\}$ is any orthonormal basis of $\mathcal{H}$ and $tr(.)$ is the natural trace on $C_1(\mathcal{H})$. The Hilbert-Schmidt norm of $X \in C_2(\mathcal{H})$ is given by $\|X\|_2 = \langle X, X\rangle^{\frac{1}{2}}$. For any operator A in $B(\mathcal{H})$, set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = |A^*|^2 - |A|^2$ (the self commutator of A), and consider the following definitions: A is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$, p-hyponormal if $|A|^{2p} \geq |A^*|^{2p}$ $(0 < p \leq 1)$, class $A(s, t)(0 < s, t \leq 1)$, if

$$(|A^*|^t|A|^{2s}|A^*|^t)^{\frac{t}{s+t}} \geq |A^*|^{2t}.$$

Especially, the class $A(1, 1)$ denote by $\mathcal{A}$ was defined first by the inequality $|A^2| \geq |A|^2$, which is equivalent to $(|A^*||A|^2|A^*|)^{\frac{1}{2}} \geq |A^*|^2$. Class $\mathcal{A}$ operators has been defined in [13] as a nice application of Furuta inequality [11]. So as a generalization of class $\mathcal{A}$, class $A(s, t)$, $(0 < s, t \leq 1)$ was defined in [10]. Inclusion relation among these classes are known as follows:

$$\{\text{class } A(s, t), \quad s, t \in (0, 1]\} \subset \{\text{class } \mathcal{A}\}.$$

An operator $A = U|A|$ is said to be *w-hyponormal*, if $|\widetilde{A}| \geq |A| \geq |(\widetilde{A})^*|$, where $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ is the Aluthge transform of A (see [1] and [2]). It is known that the class of w-hyponormal operators coincides with class $A(\frac{1}{2}, \frac{1}{2})$ (see [15, 16]).

The famous Fuglede-Putnam theorem (see [6, 14]) asserts that if A and B are normal operators and $AX = XB$ for some operator X, then $A^*X = XB^*$. Several authors have extended this theorem to non normal operators (see [6], [9], [17], [19], [18], [20]).

Berberian [5] relaxes the hypothesis on A and B by assuming A and B^* hyponormal operators and X to be Hilbert-Schmidt class. Recently, Patel [19] proved that if A and B^* are p-hyponormal such that $AX = XB$ for $X \in C_2(\mathcal{H})$, then $A^*X = XB^*$.

Let A, B in $B(\mathcal{H})$, we define the generalized derivation $\delta_{A,B}$ induced by A and B as follows:

$$\delta_{A,B}(X) = AX - XB \text{ for all } X \in B(\mathcal{H}).$$

Anderson and Foias [4] proved that if A and B are normal, S is an operator such that $AS = SB$, then

$$\|\delta_{A,B}(X) - S\| \geq \|S\|, \text{ for all } X \in B(\mathcal{H}),$$

where $\|.\|$ is the usual operator norm. Hence the range of $\delta_{A,B}$ is orthogonal to the null space of $\delta_{A,B}$. The orthogonality here is understood to be in the sense of Definition 1.2 in [3]. The purpose of this paper is to prove that the Fuglede-Putnam theorem remains true if A and B^* are class $A(t, t)(0 < t \leq 1)$. As a consequence of this result, we give a similar orthogonality result by proving that the range of the generalized derivation induced by this class of operators is orthogonal to its kernel.

2. Main Results

The basic elementary operator $M_{A,B}$ induced by the operators A and B is defined on $C_2(\mathcal{H})$ by $M_{A,B}(X) = AXB$, and the adjoint of $M_{A,B}$ is given by the formula $M^*_{A,B} = A^*XB^*$.

Proposition 2.1. *Let* $A, B \in B(\mathcal{H})$. *If* $A \geq 0$ *and* $B \geq 0$, *then* $M_{A,B} \geq 0$.

Proof. Let $X \in C_2(\mathcal{H})$,

$$\begin{aligned}\langle M_{A,B}X, X\rangle &= tr(AXBX^*)\\ &= tr(A^{\frac{1}{2}}XBX^*A^{\frac{1}{2}})\\ &= tr((A^{\frac{1}{2}}XB^{\frac{1}{2}})(A^{\frac{1}{2}}X^*B^{\frac{1}{2}})) \geq 0.\end{aligned}$$

□

Proposition 2.2. *Let* $A, B \in B(\mathcal{H})$, *then*

$$|M_{A,B}|^{\frac{1}{2^n}}(X) = |A|^{\frac{1}{2^n}}X|B^*|^{\frac{1}{2^n}},$$

$$|M^*_{A,B}|^{\frac{1}{2^n}}(X) = |A^*|^{\frac{1}{2^n}}X|B|^{\frac{1}{2^n}},$$

for each integer n.

Proof. Since $M^*_{A,B}M_{A,B}(X) = A^*AXBB^*$ and $M_{A,B}M^*_{A,B}(X) = AA^*XB^*B$ for any operator $X \in C_2(\mathcal{H})$, this yields

$$|M_{A,B}(X)| = |A|X|B^*| \text{ and } |M^*_{A,B}(X)| = |A^*|X|B|,$$

and so,

$$|M_{A,B}(X)|^{\frac{1}{2}} = |A|^{\frac{1}{2}}X|B^*|^{\frac{1}{2}} \text{ and } |M^*_{A,B}(X)|^{\frac{1}{2}} = |A^*|^{\frac{1}{2}}X|B|^{\frac{1}{2}}.$$

Consequently, we get

$|M_{A,B}(X)|^{\frac{1}{2^n}} = |A|^{\frac{1}{2^n}} X |B^*|^{\frac{1}{2^n}}$ and $|M^*_{A,B}(X)|^{\frac{1}{2^n}} = |A^*|^{\frac{1}{2^n}} X |B|^{\frac{1}{2^n}}$ for each $n \geq 0$. □

Lemma 2.3. *If* $A, B^* \in B(\mathcal{H})$ *are class* $A(t, t)$ *operators such that* $0 < t \leq 1$, *then the operator* $M_{A,B}$ *in* $C_2(\mathcal{H})$ *is also class* $A(t, t)$.

Proof. Without loss of the generality, we may assume that $t = 2^{-n}$ for some integer $n \geq 0$.

$$|M^*_{A,B}|^t |M_{A,B}|^{2t} |M^*_{A,B}|^t (X) = |A^*|^t |A|^{2t} |A^*|^t X |B|^t |B^*|^{2t} |B|^t,$$

for each $X \in C_2(\mathcal{H})$ and, therefore,

$$\begin{aligned}(|M^*_{A,B}|^t |M_{A,B}|^{2t} |M^*_{A,B}|^t)^{\frac{1}{2}}(X) &= (|A^*|^t |A|^{2t} |A^*|^t)^{\frac{1}{2}} X (|B|^t |B^*|^{2t} |B|^t)^{\frac{1}{2}} \\ &= M_{(|A^*|^t |A|^{2t} |A^*|^t)^{\frac{1}{2}}, (|B|^t |B^*|^{2t} |B|^t)^{\frac{1}{2}}}(X).\end{aligned}$$

Since

$$\begin{aligned}&(|M^*_{A,B}|^t |M_{A,B}|^{2t} |M^*_{A,B}|^t)^{\frac{1}{2}}(X) - |M^*_{A,B}|^{2t}(X) \\ &\quad = M_{(|A^*|^t |A|^{2t} |A^*|^t)^{\frac{1}{2}}, (|B|^t |B^*|^{2t} |B|^t)^{\frac{1}{2}}}(X) - M_{|A^*|^{2t}, |B|^{2t}}(X) \\ &\quad = [(|A^*|^t |A|^{2t} |A^*|^t)^{\frac{1}{2}} - |A^*|^{2t}] X [(|B|^t |B^*|^{2t} |B|^t)^{\frac{1}{2}} - |B|^{2t}] \\ &\qquad + |A^*|^{2t} X (|B|^t |B^*|^{2t} |B|^t)^{\frac{1}{2}} + (|A^*|^t |A|^{2t} |A^*|^t)^{\frac{1}{2}} |B|^{2t}.\end{aligned}$$

It follows from Proposition 2.1 that $(|M^*_{A,B}|^t |M_{A,B}|^{2t} |M^*_{A,B}|^t)^{\frac{1}{2}} \geq |M^*_{A,B}|^{2t}$, and hence $M_{A,B}$ is class $A(t, t)$. □

Lemma 2.4. *Let* $A, B^* \in B(\mathcal{H})$ *be class* $A(s, t)$ *operators such that* $0 < s, t \leq 1$ *and* $M_{A,B}$ *defined on* $C_2(\mathcal{H})$, *then* $\ker(M_{A,B} - I) \subset \ker(M^*_{A,B} - I)$, *where* I *is the identity on* $\mathcal{H}$.

Proof. Let $X \in \ker(M_{A,B} - I)$, then $(M_{A,B} - I)(X) = 0$ and $(M^2_{A,B} - I)(X) = 0$. Hence,

$$\|M_{A,B}\|_2 = \|X\|_2 \text{ and } \|M^2_{A,B}\|_2 = \|X\|_2. \tag{2.1}$$

Since A, B^* are class $A(s, t)$ operators, then from [16, Theorem 4], A, B^* are class $\mathcal{A}$ and from Lemma 2.1, we deduce that $M_{A,B}$ is class $\mathcal{A}$ in $B(C_2(\mathcal{H}))$. Thus,

$$\begin{aligned} \|M_{A,B}X\|_2^2 &= \langle |M_{A,B}|^2 X, X\rangle \\ &\leq \langle |M^2_{A,B}|X, X\rangle \\ &\leq \||M^2_{A,B}|X\|_2\|X\|_2 \\ &= \|M^2_{A,B}X\|_2\|X\|_2, \end{aligned}$$

and from (2.1), it follows that

$$\langle |M^2_{A,B}|X, X\rangle = \|X\|_2. \tag{2.2}$$

Therefore, from (2.1) and (2.2), we obtain

$$\begin{aligned} \|(|M^2_{A,B}| - I)X\|_2^2 &= \||M^2_{A,B}|X\|_2^2 - 2\langle |M^2_{A,B}|X, X\rangle + \|X\|_2^2 \\ &= \||M^2_{A,B}|X\|_2^2 - 2\|X\|_2^2 + \|X\|_2^2 \\ &= 0. \end{aligned}$$

Therefore,

$$|M_{A,B}^2|X = X. \tag{2.3}$$

On the other hand, from (2.1) and (2.3), we get

$$\left\|(|M_{A,B}^2| - |M_{A,B}|^2)^{\frac{1}{2}} X\right\|_2^2 = \langle |M_{A,B}^2|X, X\rangle - \langle |M_{A,B}|^2 X, X\rangle = 0,$$

that is,

$$(|M_{A,B}^2| - |M_{A,B}|^2)X = 0. \tag{2.4}$$

Thus, from (2.3) and (2.4), it follows that

$$(|M_{A,B}|^2 - I)X = (|M_{A,B}|^2 - |M_{A,B}^2|)X + (|M_{A,B}^2| - I)X = 0.$$

Therefore,

$$(M_{A,B}^* - I)X = (|M_{A,B}|^2 - I)X - M_{A,B}^*(M_{A,B} - I)X = 0,$$

and the proof is complete. □

Lemma 2.5. *Let* $A \in B(\mathcal{H})$, *if* A *is invertible class* $A(s, t)$ *operator, then* A^{-1} *is class* $A(s, t)$ *operator.*

Proof. From [12], we have

$$(|A^*|^t |A|^{2s} |A^*|^t)^{\frac{t}{s+t}} = |A^*|^t |A|^s (|A|^s |A^*|^{2t} |A|^s)^{\frac{-s}{s+t}} |A|^s |A^*|^t. \tag{2.5}$$

Since A is class $A(s, t)$ operator, then $(|A^*|^t |A|^{2s} |A^*|^t)^{\frac{t}{s+t}} \geq |A^*|^{2t}$, and so

$$|A^*|^{-t} (|A^*|^t |A|^{2s} |A^*|^t)^{\frac{t}{s+t}} |A^*|^{-t} \geq I. \tag{2.6}$$

Therefore, from (2.5) and (2.6), we get

$$|A|^s (|A|^s |A^*|^{2t} |A|^s)^{\frac{-s}{s+t}} |A|^s \geq I.$$

Hence

$$(|(A^{-1})^*|^s |A^{-1}|^{2t} |(A^{-1})^*|^s)^{\frac{s}{s+t}} \geq |(A^{-1})^*|^s.$$

Therefore A^{-1} is class $A(s, t)$ operator. □

Theorem 2.6. *If A is class $A(t, t)$ operator and B^* is invertible class $A(t, t)$ operator such that $AX = XB$ for some X in $C_2(\mathcal{H})$, then $A^*X = XB^*$.*

Proof. Let $AX = XB$ for some X in $C_2(\mathcal{H})$, then $M_{A, B^{-1}}(X) = X$. Since B^* is invertible class $A(t, t)$ operator, then by Lemma 2.3, $(B^*)^{-1}$ is class $A(t, t)$ operator and by Lemma 2.1, $M_{A, B^{-1}}$ is also class $A(t, t)$ operator. Hence by lemma $M^*_{A, B^{-1}}(X) = X$, that is, $A^*X = XB^*$, by this, we complete the proof. □

Corollary 2.7 ([14])**.** *If A is w-hyponormal and B^* is invertible w-hyponormal operator such that $AX = XB$ for some X in $C_2(\mathcal{H})$, then $A^*X = XB^*$.*

Corollary 2.8. *If A is class $\mathcal{A}$ and B^* is invertible class $\mathcal{A}$ such that $AX = XB$ for some X in $C_2(\mathcal{H})$, then $A^*X = XB^*$.*

Now, we ready to extend the orthogonality result to the class $A(t, t)$ operators.

Theorem 2.9. *Let A, B be operators in $B(\mathcal{H})$ and $S \in C_2(\mathcal{H})$. Then*

$$\|\delta_{A, B}(X) + S\|_2^2 = \|\delta_{A, B}(X)\|_2^2 + \|S\|_2^2, \tag{2.7}$$

and

$$\|\delta^*_{A, B}(X) + S\|_2^2 = \|\delta^*_{A, B}(X)\|_2^2 + \|S\|_2^2, \tag{2.8}$$

if and only if $\delta_{A, B}(S) = 0 = \delta_{A^*, B^*}(S)$ for all $S \in C_2(\mathcal{H})$.

Proof. We known that the Hilbert-Schmidt class $C_2(\mathcal{H})$ is a Hilbert space. Note that

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2 + \mathrm{Re}\langle \delta_{A,B}(X),\ S\rangle$$

$$= \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2 + \mathrm{Re}\langle X,\ \delta^*_{A,B}(S)\rangle,$$

and

$$\|\delta^*_{A,B}(X) + S\|_2^2 = \|\delta^*_{A,B}(X)\|_2^2 + \|S\|_2^2 + \mathrm{Re}\langle X,\ \delta_{A,B}(S)\rangle.$$

Hence by the equality $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$, we obtain (2.7) and (2.8).

□

The claim is verified and the proof is complete.

Corollary 2.10. *Let* A, B *be operators in* $B(\mathcal{H})$ *and* $S \in C_2(\mathcal{H})$. *Then*

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2,$$

and

$$\|\delta^*_{A,B}(X) + S\|_2^2 = \|\delta^*_{A,B}(X)\|_2^2 + \|S\|_2^2,$$

if and only if either of the following hypothesis hold:

(1) *A is class $A(t, t)$ operator and B^* is invertible class $A(t, t)$ operator*.

(2) *A is w-hyponormal operator and B^* is invertible w-hyponormal operator*.

(3) *A is class $\mathcal{A}$ and B^* is invertible class $\mathcal{A}$.*

References

[1] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Integral Equations Operator Theory 13 (1990), 307-315.

[2] A. Aluthge and D. Wang, w-Hyponormal operators, Integral Equations Operator Theory 36 (2000), 1-10.

[3] J. H. Anderson, On normal derivation, Proc. Amer. Math. Soc. 38 (1973), 136-140.

[4] J. H. Anderson and C. Foias, Properties which normal normal operators share with normal derivations and related operators, Pacific J. Math. 61 (1975), 313-325.

[5] S. K. Berberian, Extensions of theorem of Fuglede and Putnam, Proc. Amer. Math. Soc. 71(1) (1978), 113-114.

[6] Hyung Koo Cha, An extension of the Fuglede-Putnam theorem to w-hyponormal operators, J. Korea Soc. Math. Edu. Ser. B 10 (2003), 273-277.

[7] M. Cho and T. Huruya, p-Hyponormal operators for $0 < p < \frac{1}{2}$, Comment. Math. Prace Math. 33 (1993), 23-29.

[8] J. B. Conway, Subnormal Operators, Research Notes in Mathematics, 51, Pitman (Advanced Publishing Program), Boston, Mass-London, 1981.

[9] B. P. Duggal, On dominant operators, Arch. Math. 46(4) (1986), 353-359.

[10] M. Fujii, D. Jung, S. H. Lee, M. Y. Lee and R. Nakamoto, Some classes of operators related to paranormal and log-hyponormal operators, Math. Japan 51 (2000), 395-402.

[11] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{(p+2r)}{q}}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p + 2r$, Proc. Amer. Math. Soc. 101 (1987), 85-88.

[12] T. Furuta, Extension of the Furuta inequality and Ando-Hiai log-majorization, Linear Algebra Appl. 219 (1995), 139-155.

[13] T. Furuta, M. Ito and T. Yamazaki, A subclass of paranormal operators including class of log-hyponormal and several related classes, Sci. Math. 1 (1998), 389-403.

[14] P. R. Halmos, A Hilbert Space Problem Book, Second Edition, Graduate Texts in Mathematics, Encyclopedia of Mathematics and its Applications, 17, Springer-Verlag, New York-Berlin, 1982.

[15] M. Ito, Some classes of operators associated with generalized Aluthge transformation, SUT J. Math. 35 (1999), 149-165.

[16] M. Ito and T. Yamazaki, Relation between two inequalities $\left(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq B^{r}$ and $A^{p} \geq \left(A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}}\right)^{\frac{p}{p+r}}$ and their applications, Integral Equations Operator Theory 44 (2002), 442-450.

[17] R. L. Moore, D. D. Rogers and T. T. Trent, A note on intertwining M-hyponormal operators, Proc. Amer. Math. Soc. 83 (1961), 514-516.

[18] M. Radjabalipour, An extension of Putnam-Fuglede theorem for hyponormal operators, Math. Z 194 (1987), 117-120.

[19] S. M. Patel, On intertwining p-hyponormal operators, Indian J. Math. 38 (1996), 287-290.

[20] J. G. Stampi and B. Wadhwa, An asymmetric Putnam-Fuglede theorem for dominant operators , Indiana Univ. Math. J. 25 (1976), 359-365.

■